# Collective Motion Controll on Networks of Planar Robots

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## I. INTRODUCTION

Network robotic systems enjoy an incredibly broad variety of applications, both scientific and commercial. One prominent example arises in oceanography, with the need for groups of Autonomous Underwater Vehicles (AUVs) [2]. Other examples arise in commercial fields such as the automotive industry, with the need for communication between self-driving cars [3].

The controllers put forth by Sepulchre et al. have applications in network robotics, where collective motions can be required to synchronize measurements on certain time and length scales. The limited communication of agents in harsh environments inspired the study of these controllers with a more flexible communication network [2].

In this review, seek to provide a succinct yet thorough overview of the motion controllers explored by Sepulchre et al. in their papers on collective motions of a planar particle model with unrestricted all-to-all communication between agents. We offer a high level summary of the controllers, and sketches of some proofs related to their convergence properties.

## II. NOTATION

### A. Agent Model

In this report, we consider a dynamical system of N identical agents, all of unit mass. The column vector $r \in \mathbb{C}^N$ denotes the agents' positions ($r_k$ refers to the position of agent $k$, where $r_k = x_k + iy_k$). The column vector $\theta \in S^N$ denotes the agents' headings ($\theta_k$ refers to the heading angle of agent $k$ measured counter clockwise from the positive x axis). $r_k$ fully defines agent $k$'s position, while $\theta_k$ defines agent $k$'s heading, as shown in Fig. 1.

The center of mass for the system is denoted $R \in \mathbb{C}$. Note that, as all agents are identical, $R$ is simply the average of the elements in $r$.

All the agents in the network experience a steering controller denoted by the vector $\boldsymbol{u} \in S^N$, where for each agent k, $\dot{\theta_k} = u_k$.

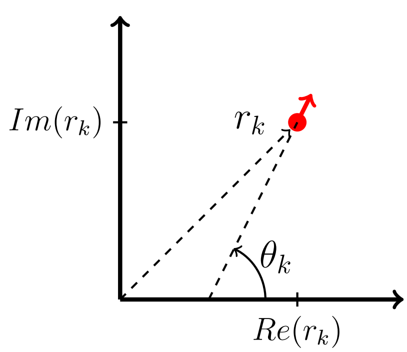



FIG. 1: Position and heading of agent $k$ in relation to $r_k$ and $\theta_k$.

### B. Inner Product

Throughout this report, we will be utilizing a standard inner product, which is denoted by $\langle \cdot, \cdot \rangle$. Given complex numbers $z_1, z_2 \in \mathbb{C}$, the inner product $\langle z_1, z_1 \rangle = Re(\bar{z}_1 z_2)$. If instead given the complex column vectors $z, w \in \mathbb{C}^N$, the inner product becomes $\langle z, w \rangle = Re(\bar{z}^T w)$.

## III. CONTENT

### A. Steered Agent Model

In this report, we consider the model defined below. Every agent moves at unit speed, and is only influenced by the steering controller $u$.

$$\dot{r}_k = e^{i\theta_k} = \cos(\theta_k) + i\sin(\theta_k) \tag{1}$$

$$\dot{\theta}_k = u_k \tag{2}$$

## IV. HIGH LEVEL SUMMARY OF CONTROLLERS

### A. Controller 1: Group Linear Momentum

To have the most fundamental notion of control over the agents, we propose a controller that minimizes or maximizes the total linear momentum. The total linear momentum is minimized when all the agents in the system travel in a fashion such that the location of the center of mass does not change. We call this a balanced state. The total linear momentum is maximized when all the agents are traveling in the same direction. We call this a synchronized (or synced) state.

To create a controller that balances or syncs the agents, we start by defining $p_\theta$ below in Eq. 3. This quantity represents the velocity of the center of mass of all agents, and is directly related to the net linear momentum of the system. We also define our potential function $U_1$ in Eq. 4.

$$p_\theta = \dot{R} = \frac{1}{N}\sum_{k=1}^{N} \dot{r_k} = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k} \tag{3}$$

$$U_1(\boldsymbol{\theta}) = \frac{N}{2}|p_\theta|^2 \tag{4}$$

Using the negative gradient controller shown below in Eq. 5, $U_1$ will be brought to one of its critical points.

$$u_k = -K\frac{\partial U_1}{\partial \theta_k} \tag{5}$$

To give some insight into the critical points and their stability properties, we explore our expression for $U_1$. Note that the critical points occur when $\frac{\partial U_1}{\partial \theta_k} = 0\ \forall k$. Two can be accounted for by intuition.

One is the global minimum with $|p_\theta| = 0$. This occurs when all the agents are balanced, as defined at the beginning of this section. This state is locally asymptotically stable when $K > 0$, and unstable otherwise.

The other intuitive critical point is the global maximum of $U_1$, corresponding to a synchronized state with all agents moving with the same heading. This state is locally asymptotically stable when $K < 0$, and unstable otherwise.

To find other critical points, we inspect our expression for the partial derivative.

$$\frac{\partial U_1}{\partial \theta_k} = N\langle p_\theta, \frac{\partial p_\theta}{\partial \theta_k}\rangle \tag{6}$$

We see

$$\frac{\partial p_\theta}{\partial \theta_k} = \frac{\partial}{\partial \theta_k}(\frac{1}{N}\sum_{k=1} N e^{i\theta_k}) = \frac{1}{N} i e^{i\theta_k} \tag{7}$$

Let $p_\theta = |p_\theta| e^{i\psi}$, so that for all critical points we have

$$0 = \frac{\partial U_1}{\partial \theta_k} = |p_\theta|\langle e^{-i\psi}, ie^{i\theta_k}\rangle \tag{8}$$

This leads to the condition

$$0 = \sin(\theta_k - \psi) \tag{9}$$

We can only have agents with heading $\psi$ or the an opposite heading $\psi + \pi \mod 2\pi$, with the majority of the agents having the former heading. Let $0 \leq M < \frac{N}{2}$ be the number of agents with heading $\psi + \pi \mod 2\pi$. Then we see

$$|p_\theta| = |\frac{N-M}{N} e^{i\psi} + \frac{M}{N} e^{i(\psi+\pi)}| = 1 - \frac{2M}{N} \geq \frac{1}{N}. \tag{10}$$

The last inequality comes from scenario when all but one agents have opposite headings, leading to the minimum non-zero average momentum $|p_\theta| = \frac{1}{N}$.

If $M = 0$, we are in the synchronized state, discussed at the beginning of this section. If $M \neq 0$, we then have

$$\frac{\partial^2 U_1}{\partial {\theta_k}^2} = \frac{1}{N} - |p_\theta| \cos(\psi - \theta_k) \tag{11}$$

This expression is positive if $\theta_k = \psi + \pi$, and negative if $\theta_k = \psi$. Thus, these arrangements are unstable for all $K \neq 0$, as they are saddle nodes.

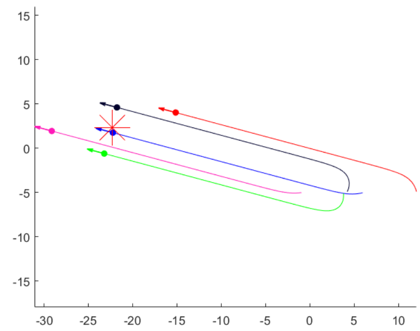
With the steering controller described in Eq. 5, all agents will converge to a linear path, where the agents are synced if $K < 0$ and balanced if $K > 0$. We now define the new steering controller shown in Eq. 12 which we will call Controller 1. Controller 1 can be thought of as the controller in Eq. 5, but in a rotating reference frame. What results is the same convergence to a synced state when $K < 0$, and to a balanced state when $K > 0$. However, if $\omega_0 \neq 0$, the agents travel in circular paths. Simulations with this controller are presented in Fig. 2.

$$u_k = \omega_0 - K\frac{\partial U_1}{\partial \theta_k} \tag{12}$$

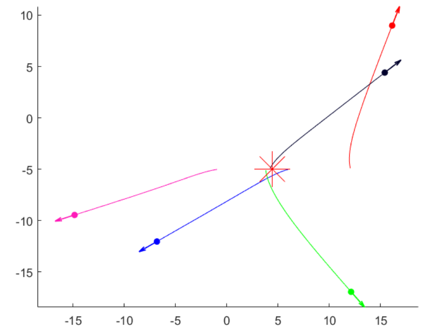
### B. Controller 2: Common Center of Orbit

The first step toward having complete control over the network has been taken, although the previous controller has very limited applications in the real world. One lacking feature is a common center of orbit for all agents.

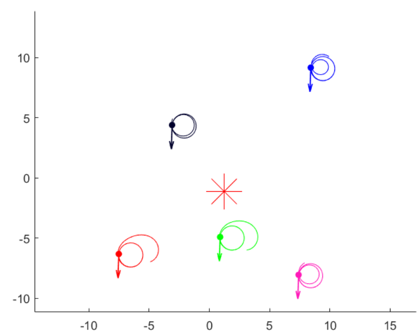
We would now like to motivate the design of a controller that allows all the agents to travel along circular paths around the same center of orbit. To do this, we will define several useful quantities, and give a sketch of the analysis of a suitable Lyapunov function.



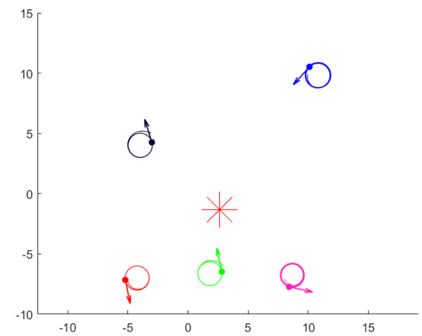
(a) Synced agents traveling in a linear path with $w_0 = 0$, and $K = -1$



(b) Balanced agents traveling in a linear path $w_0 = 0$, and $K = 1$



(c) Synced agents traveling in cirular paths with $w_0 = 1$, and $K = -1$



(d) Balanced agents traveling in circular paths with $w_0 = 1$, and $K = 1$

FIG. 2: Simulation using Controller 1, defined in Eq. 12, to influence agents be synced or balanced with linear or circular paths. For each agent, the corresponding colored dot represents its location, and the attached arrow represents its heading. The red star represents the center of mass of the system.

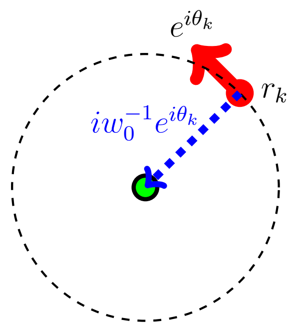



FIG. 3: Circular orbit of agent k traveling with an angular velocity $\omega_0$. Since the radius of the circle is $|\omega_0|^{-1}$, the vector from the agent to the center is $i\omega_0^{-1}e^{i\theta_k}$.

For an agent $k$ moving with some constant steering control $\omega_0$, the center of its circular orbit is $c_k = r_k + i\omega_0^{-1}e^{i\theta_k}$. This can be seen in Fig. 3. For convenience, let $s_k = -i\omega_0 c_k = e^{i\theta_k} - i\omega_0 r_k$. Also, let $P = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T$.

Note that, for any vector $\boldsymbol{u}$, we have $P\boldsymbol{u} = \boldsymbol{u} - average(\boldsymbol{u})\mathbb{1}_N$. When all agents orbit a common center, $s_k = \xi\ \forall k$ for some $\xi \in \mathbb{C}$, so $P\boldsymbol{s} = \xi\mathbb{1}_N - \xi\mathbb{1}_N = \mathbb{0}_N$. We will therefore design a controller to minimize the quantity $P\boldsymbol{s}$. This can be done by considering the Lyapunov function $S(\mathbf{r},\boldsymbol{\theta})$ defined below.

$$S(\mathbf{r},\boldsymbol{\theta}) = \frac{1}{2}\|P\mathbf{s}\|^2 \tag{13}$$

Along a trajectory, the time derivative of $S$ is

$$\dot{S} = \langle P\mathbf{s}, P\dot{\mathbf{s}}\rangle = \langle P^2\mathbf{s}, \dot{\mathbf{s}}\rangle = \langle P\mathbf{s}, \dot{\mathbf{s}}\rangle. \tag{14}$$

This comes from two properties of $P$: $P$ is Hermitian matrix, since $P = P^T$, and $P \in \mathbb{R}^{N\times N}$, and $P$ is a projection matrix, so $P^2 = P$. Writing $\dot{S}$ as a summation of its components, we have

$$\dot{S} = \sum_{k=0}^{N} \langle P_k\mathbf{s}, ie^{i\theta_k}\rangle (u_k - \omega_0) \tag{15}$$

In order to use the LaSalle invariance principle, we must have a negative semidefinite Lie derivative of $S$. We can accomplish this by choosing a controller such that $\dot{S}$ simplifies to a negative constant multiplied by a squared quantity: $u_k = \omega_0 - \kappa\langle P_k\mathbf{s}, ie^{i\theta_k}\rangle$ . Then

$$\dot{S} = -\kappa \sum_{k=0}^{N} \langle P_k\mathbf{s}, ie^{i\theta_k}\rangle^2. \tag{16}$$

We see that $\dot{S} = 0$ when $\mathbf{s} = \xi\mathbb{1}_N$, as desired. We shall omit the calculation, but use the result $\langle P_k\mathbf{s}, ie^{i\theta_k}\rangle = -\langle \omega_0\tilde{r}_k, e^{i\theta_k}\rangle - \frac{\partial U}{\partial \theta_k}$ where $\tilde{r}_k = r_k - R$. Details can be found in Sepulchre et al. The controller then takes the form

$$u_k = \kappa\frac{\partial U_1}{\partial \theta_k} + \omega_0(1 + \kappa\langle \tilde{r}_k, \dot{r}_k\rangle) \tag{17}$$

where $\omega_0 \neq 0$ and $\kappa > 0$. In this report, we refer to the steering controller in Eq. 17 as Controller 2. A little about convergence: $S$ is positive definite and proper in the space we consider, and $\dot{S}$ is negative definite about $P\boldsymbol{s} = \mathbb{0}_N$, so the LaSalle invariance principle guarantees convergence to the largest invariant set with $\dot{S} = 0$. This set is characterized by $\langle P_k\mathbf{s}, ie^{i\theta_k}\rangle = 0\ \forall k$. With this condition, $u_k = \omega_0$, and $\boldsymbol{s} = \xi\mathbb{1}_N$.

Therefore, with the controller defined by 17, we can bring all agents into a circular orbit around the same center. One will note, however, that the phase arrangement of the agents has no bearing on the equilibrium state. A simulation of a system under the influence of Controller 2 is shown in Fig. 4.

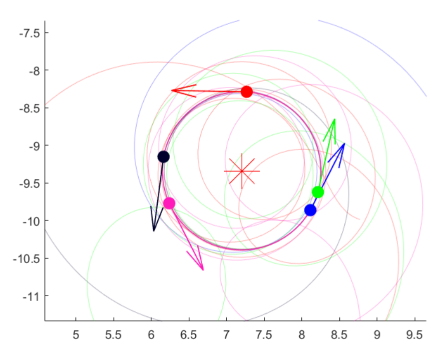



FIG. 4: Simulation using Controller 2 defined in Eq. 17 to influence all agents to orbit the same center. For this simulation, $w_0 = 1$ and $k = 1$.

### C. Controller 3: Specific Phase Arrangements

To extend the capabilities of Controller 2, which allowed all the agents to orbit the same center, we present Controller 3. Controller 3 adds the ability to determine the phase arrangements, or spacing of agents along the circular path.

In order to provide this additional control feature, we introduce an arbitrary smooth potential function $U(\boldsymbol{\theta})$ with $\langle \nabla U, \mathbb{1}_N \rangle = 0$. With the steering controller defined in Eq. 18, the network will converge to a configuration where all agents travel along the same circular path, and the phase configuration is such that the potential function $U$ is at a local minimum.

$$u_k = \omega_0(1 + \kappa\langle \tilde{r}_k, \dot{r}_k \rangle) - \frac{\partial(U - \kappa \partial U_1)}{\partial \theta_k} \tag{18}$$

Above, $\tilde{r}_k = r_k - R$, $\omega_0 \neq 0$, and $\kappa > 0$. To give some insight, we will sketch the analysis of the Lyapunov function defined below:

$$V(\mathbf{r}, \boldsymbol{\theta}) = \kappa S(\mathbf{r}, \boldsymbol{\theta}) + U(\boldsymbol{\theta}). \tag{19}$$

Along a trajectory, the time derivative of $V$ is given by

$$\dot{V} = \kappa \dot{S} + \dot{U} = \sum_{k=0}^{N} \kappa \langle P_k \mathbf{s}, ie^{i\theta_k} \rangle (u_k - \omega_0) + \frac{\partial U}{\partial \theta_k} \dot{\theta}_k \tag{20}$$

The property $\langle \nabla U, \mathbb{1}_N \rangle = 0$ demanded of $U$ now proves helpful: let us add $-\omega_0 \sum_{k=0}^{N} \frac{\partial U}{\partial \theta_k} = 0$.

$$\dot{V} = \sum_{k=0}^{N} (\kappa \langle P_k \mathbf{s}, ie^{i\theta_k} \rangle + \frac{\partial U}{\partial \theta_k})(u_k - \omega_0) \tag{21}$$

We can now substitute a useful form for our controller in Eq 18:

$$u_k = \omega_0 - \kappa \langle P_k \boldsymbol{s}, ie^{i\theta_k} \rangle - \frac{\partial U}{\partial \theta_k}. \tag{22}$$

This gives us an expression that is easy to analyze:

$$\dot{V} = -\sum_{k=0}^{N} (u_k - \omega_0)^2 \leq 0. \tag{23}$$

Since $V$ has a lower bound and $\dot{V}$ is negative definite about the state when $u_k = \omega_0\ \forall k$, trajectories will converge to the largest invariant set with $\kappa \langle P_k \boldsymbol{s}, ie^{i\theta_k} \rangle = -\frac{\partial U}{\partial \theta_k}$. Since $u_k = \omega_0\ \forall k$ in this set, we have constant $U$, so $\kappa \langle P_k \boldsymbol{s}, ie^{i\theta_k} \rangle = 0$. We then arrive at $\boldsymbol{s} = \xi \mathbb{1}_N$, so we are again in a state with all agents orbiting a common center. In addition to synchronizing the center of all agents' orbits, we now have the critical set of $U$ to guarantee desired phase arrangements. Stability analysis of the critical set can be found in Sepulchre et al.

Since the network converges to a phase configuration such that $U$ reaches a local minimum, one simply needs to design the arbitrary potential function $U$ so that its only local minimum is where $\theta$ is in the desired phase configuration. We now give a few different examples of possible potential functions $U$ that can lead to desired phase arrangmeents.

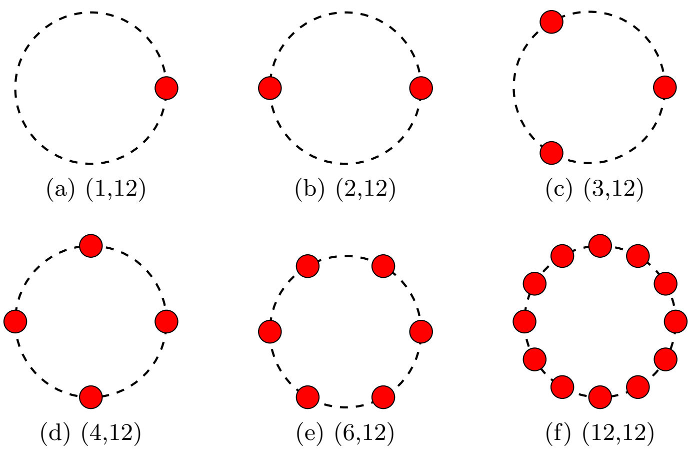



FIG. 5: Example (M,N) Patterns with N = 12. In the figures above, each red dot represents a cluster of agents. Each cluster on the same circle contains the same number of agents (N/M).

#### 1. (M,N) Pattern on a Circular Path

Given a network of N agents traveling along the same circular path, let M be an integer divisor of N. If the N agents travel along the circle such that they are split among M evenly spaced clusters, their configuration is described as an (M,N) pattern. Each cluster along the circular path contains (N/M) agents, all located at the same point traveling in the same direction. For M = 1, all the agents are located at the same point and travel along the same trajectory. For M = N, all the agents are evenly spaced along the circular path. Example (M,N) patterns with N = 12 are shown in Fig. 5.

#### 2. Potential Function U for (M,N) Pattern

To achieve circular formations around a common center with $(M, N)$-pattern phase arrangements, we use the controller defined in Eq. 18 with the following potential $U^{M,N}$. A simulation result is presented in Fig. 6. Analysis of this controller can be found in Sepulchre et al.

$$U^{M,N} = \sum_{m=1}^{M} K_m U_m, \qquad \begin{cases} K_m > 0 & m \in \{1, ...M\} \\ K_m < 0 & m = M \end{cases} \tag{24}$$

where

$$U_m = \frac{N}{2}|p_{m\theta}|^2, \quad \text{and} \quad |p_{m\theta}| = \frac{1}{mN}\sum_{k=1}^{N} e^{im\theta_k}. \tag{25}$$

### D. Extensions of Controller 3

After convergence under Controller 3, all agents travel along the same circular path with a certain phase arrangement. However, there are certain properties that are not determined by Controller 3, and only depend on the initial position and heading of the agents. In this section, we introduce some modifications to Controller 3 to add additional features.

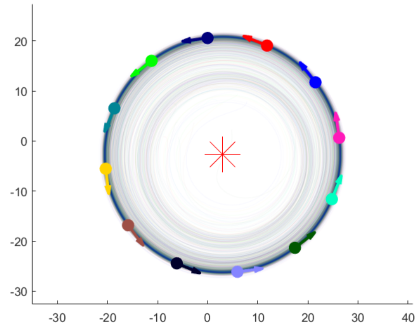



FIG. 6: Simulation using Controller 3 defined in 18.

### *1. Circular Paths Around a Beacon*

With Controller 3, all the agents orbit the same center, however the exact location of this center cannot be controlled. The controller defined below allows the center of the agents' orbits to be specified. Suppose we would like the orbits to be around a location $R_0$. Letting, $\tilde{r}_k = r_k - R_0$, we can use the controller defined in Eq. 18 with $U_1$ removed.

$$u_k = \omega_0(1 + \kappa\langle r_k - R_0, \dot{r}_k\rangle) - \frac{\partial U}{\partial \theta_k} \tag{26}$$

This controller was inspired by considering a Lyapunov function that was similar to the function presented in Eq. 19, but with an addition term. This additional term is minimized when the center of the circular path is at $R_0$. Analysis provided in Sepulchre et al. leads to the result that the system converges to equilibrium states with phase arrangements in the critical set if $U$ as well as all orbits around $R_0$.

### *2. Phase timing*

After convergence under Controller 3, the phase of agent k as a function of time will be $\theta_k(t) = \omega_0(t - t_{k0})$ mod $2\pi$. All agents will be orbiting a center at the same angular velocity, $\omega_0$, in a fixed phase arrangement. However, the value $t_{k0}$ that corresponds to the exact timing of each agent's phase cannot be influenced. A slight modification can be made to Controller 3 to add this additional capability. For $1 < k < N - 1$, the modified controller is the same as Controller 3. The steering control for the $N^{th}$ agent is replaced by

$$u_N = \omega_0(1 + \kappa\langle \tilde{r}_N, \dot{r}_N\rangle) - \frac{\partial(U - \kappa\partial U_1)}{\partial \theta_N} + d\sin(\theta_0 - \theta_N(t)) \tag{27}$$

where $\theta_N(t)$ is the desired phase for agent N as a function of time, such that $\dot{\theta}_N = \omega_0$. This function essentially fixes the phase of agent N as a function of time, and therefore influences the offset of the entire phase arrangement. A proof of the convergence properties of this controller can be found in Sepulchre et al.

## V. SUMMARY

In order to study the collective motions of robots in a network, we considered a system of N identical agents, each moving at unit speed, and each with a steering controller. We analyzed several of the steering controllers proposed in Sepulchre et al. Each of these controllers was inspired by defining a Lyapunov function that reached a minimum at the desired configuration. Controllers were chosen such that the

Lie derivative of each Lyapunov potential function was negative semi-definite, resulting in at least local convergence to the desired configuration.

In Controller 1, we developed the capability to sync or balance agents that are traveling in either straight or circular paths. In Controller 2, we added the feature to guide all agents to circular orbits around a common center. With Controller 3, we were able to add the feature of influencing the phase arrangements of the agents along their common circular orbit. Lastly, we modified Controller 3 so that the location of the orbits' centers and phase offset of the agents could be specified. With the rich features provided by these controllers, agents in a planar robotic network can partake in desirable collective movements for a variety of applications.

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